

EARLY TWO-DIMENSIONAL RECONSTRUCTION AND RECENT TOPICS STEMMING FROM IT

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by

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In 1955 I was a Lecturer in Physics at the University of Cape Town when the Hospital Physicist at the Groote Schuur Hospital resigned. South African law required that a properly qualified physicist supervise the use of any radioactive isotopes, and since I was the only nuclear physicist in Cape Town, I was asked to spend 1 1/2 days a week at the hospital attending to the use of isotopes, and I did so for the first half of 1956. I was placed in the Radiology Department under Dr. J. Muir Grieve, and in the course of my work I observed the planning of radiotherapy treatments. A girl would superpose isodose charts and come up with isodose contours which the physician would then examine and adjust, and the process would be repeated until a satisfactory dose-distribution was found. The isodose charts were for homogeneous materials, and it occurred to me that since the human body is quite inhomogeneous these results would be quite distorted by the inhomogeneities - a fact that physicians were, of course, well aware of. It occurred to me that in order to improve treatment planning one had to know the distribution of the attenuation coefficient of tissues in the body, and that this distribution had to be found by measurements made external to the body. It soon occurred to me that this information would be useful for diagnostic purposes and would constitute a tomogram or series of tomograms, though I did not learn the word "tomogram" for many years.

At that time the exponential attenuation of X- and gamma-rays had been known and used for over sixty years with parallel sided homogeneous slabs of material. I assumed that the generalization to inhomogeneous materials had been made in those sixty years, but a search of the pertinent literature did not reveal that it had been done, so I was forced to look at the problem ab initio. It was immediately evident that the problem was a mathematical one which can be seen from Fig. 1. If a fine beam of gamma-rays of intensity I_0 is incident on the body and the emerging intensity is I , then the measurable quantity $g = \ln(I_0/I) = \int_L f ds$, where f is the variable absorption coefficient along the line L . Hence if f is a function in two dimensions, and g is known for all lines intersecting the body, the question is: "Can f be determined if g is known?". Again this seemed like a problem which would

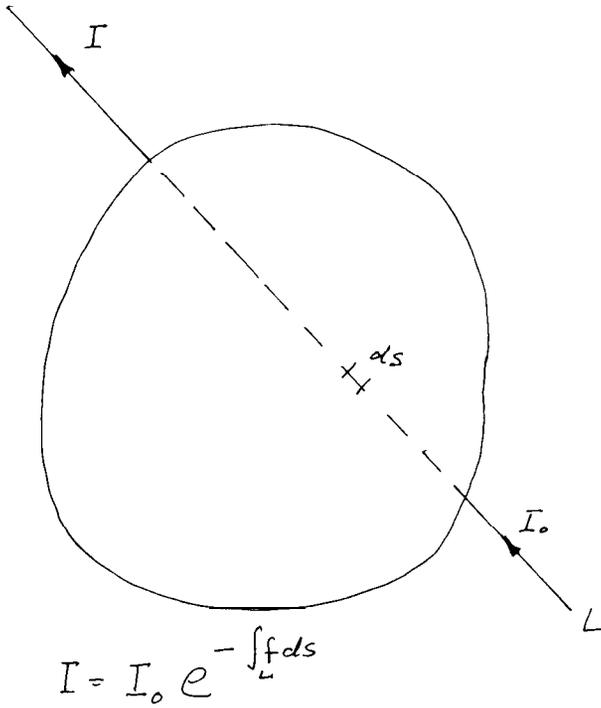


Fig. 1

have been solved before, probably in the 19th Century, but again a literature search and enquiries of mathematicians provided no information about it. Fourteen years would elapse before I learned that Radon had solved this problem in 1917. Again I had to tackle the problem from the beginning. The solution is easy for objects with circular symmetry for which $f = f(r)$, r being the radius. One has Abel's equation to solve, and its solution

$$f(r) = -\frac{d}{dr} \left[\frac{r}{\pi} \int_r^\infty \frac{g(s) ds}{s \sqrt{s^2 - r^2}} \right] = -\frac{dI(r)}{dr}, \tag{1}$$

has been known since 1825. In 1957 I did an experiment in Cape Town on a circularly symmetrical sample consisting of a cylinder of aluminum surrounded by an annulus of wood. The results are shown in Fig. 2. Here $I(r)$ is plotted against r and the constant slopes indicate the constant values of the absorption coefficient in wood and aluminum. Even this simple result proved to have some predictive value for it will be seen that the three points nearest the origin lie on a line of a slightly different slope from the other points in the aluminium. Subsequent inquiry in the machine shop revealed that the aluminum cylinder contained an inner peg of slightly lower absorption coefficient than the rest of the cylinder.

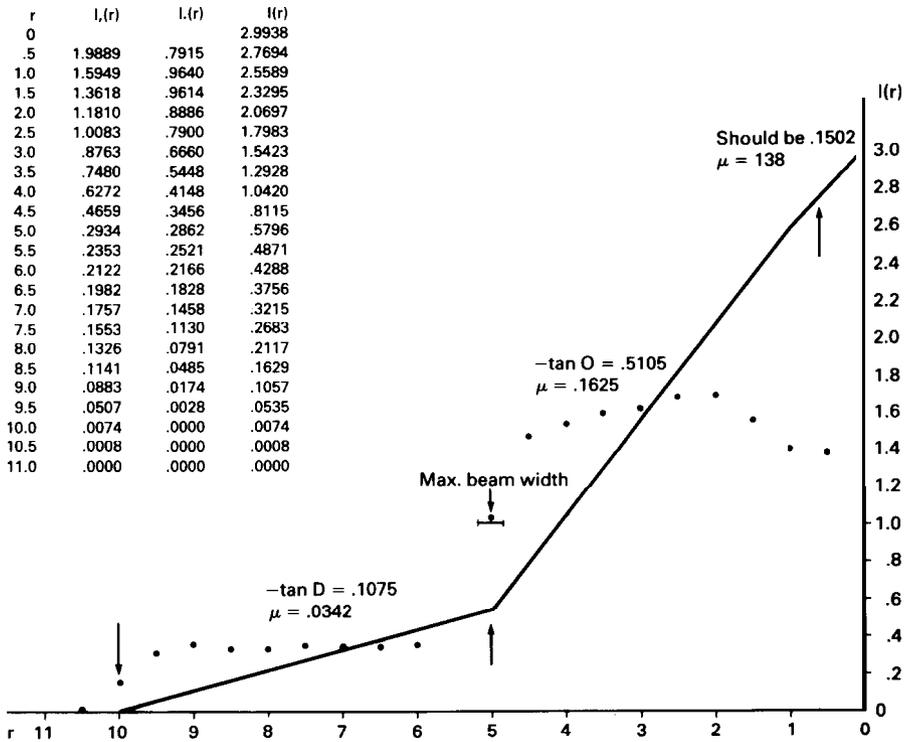


Fig. 2

Further work occurred intermittently over the next six years. Using Fourier expansions of f and g , I obtained equations like Abel's integral equation but with more complicated kernels, and I obtained results like equation (1), but with more complicated integrands. However, they were also integrals from r to ∞ , a point which I shall return to. These integrals were not too good for data containing noise, so alternative expansions were developed which dealt with noisy data satisfactorily. By 1963 I was ready to do an experiment on a phantom without circular symmetry with the apparatus shown in Fig. 3. The two cylinders are two collimators which contain the source and the detector, and which permit a 5mm beam of Co^{60} gamma-rays to pass through the phantom which is the disc in between them. At that time I was approached by an undergraduate, David Hennessee, who wanted to know whether I had a numerical problem for him to bark on so that he could learn FORTRAN and learn how to use a computer. Since he had not had experience of dealing with data in which the principal source of error was statistical, this seemed a good chance for him to learn that too by helping take the data. Final data were taken over two days in the summer of 1963, the calculations were done, and the results are shown in Figs. 4 a and b. Figure 4 a shows the phantom, and

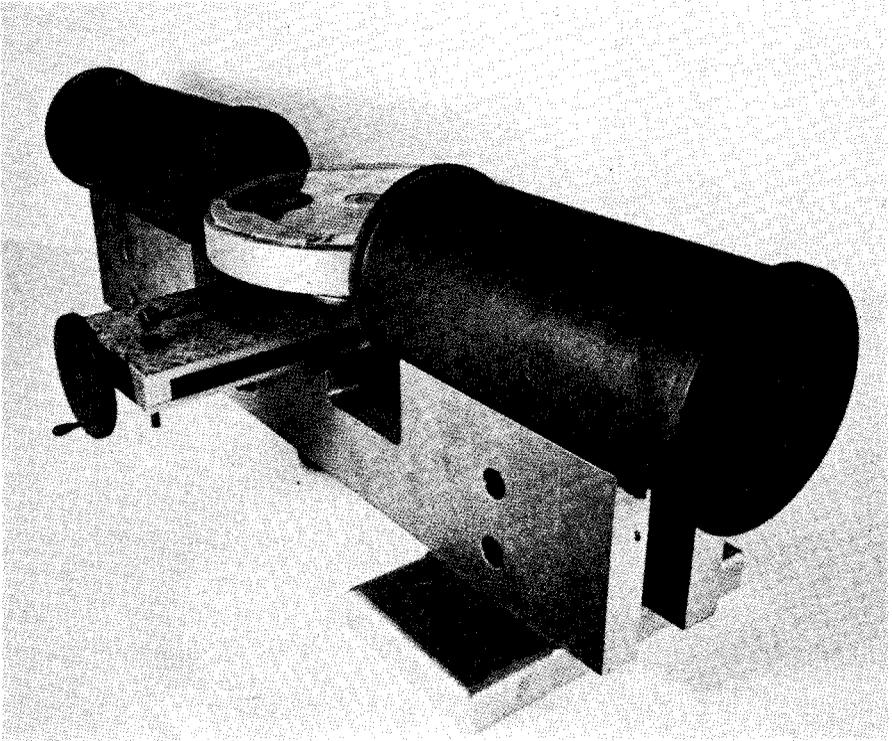


Fig. 3

this was quite a hybrid. The outer ring of aluminum represents the skull, the Lucite inside it represents soft tissue, and the two aluminum discs represent tumors. The ratio of the absorption coefficients of aluminum and lucite is about 3, which is very much greater than the ratio of the absorption coefficients of abnormal and normal tissue. The ratio of 3 was chosen in an attempt to attract the attention of those doing what would now be called emission scanning using positron emitting isotopes, a subject then in its infancy. This was about the ratio between the concentration of radioisotope in abnormal tissue and muscle on the one hand and in normal tissue on the other. The mathematics of emission scanning is of course the same as transmission scanning after an obvious correction for absorption. The right side of the slide shows the results plotted as a graph of attenuation coefficient as a function of distance along the line OA. Similar graphs were made along other lines. The full curves are the true values, the dots are the calculated values, and the agreement is quite good. The results could have been presented on a grey scale on a scope, but the graphs were thought to be better for publication.

Publication took place in 1963 and 1964 (1, 2). There was virtually no response. The most interesting request for a reprint came from a Swiss Centre for Avalanche Research. The method would work for deposits of

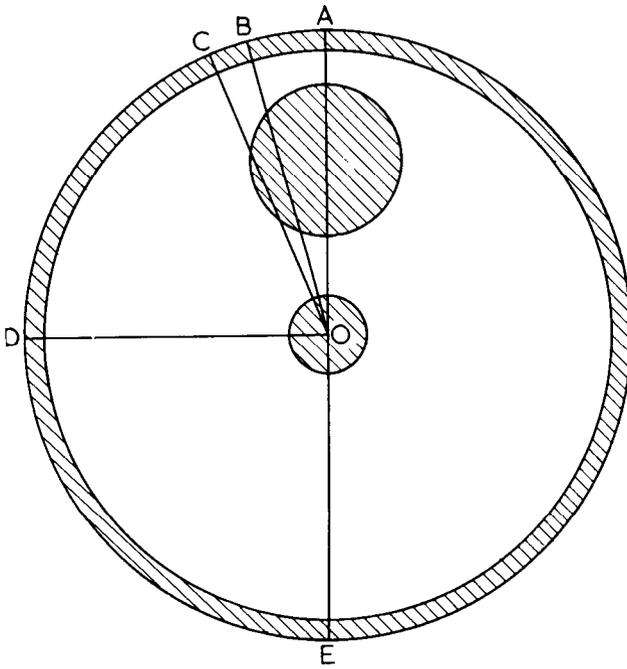


Fig. 4 a

snow on mountains if one could get either the detector or the source into the mountain under the snow!

My normal teaching and research kept me busy enough so I thought very little about the subject until the early seventies. Only then did I learn of Radon's (3) work on the line integral problem. About the same time I

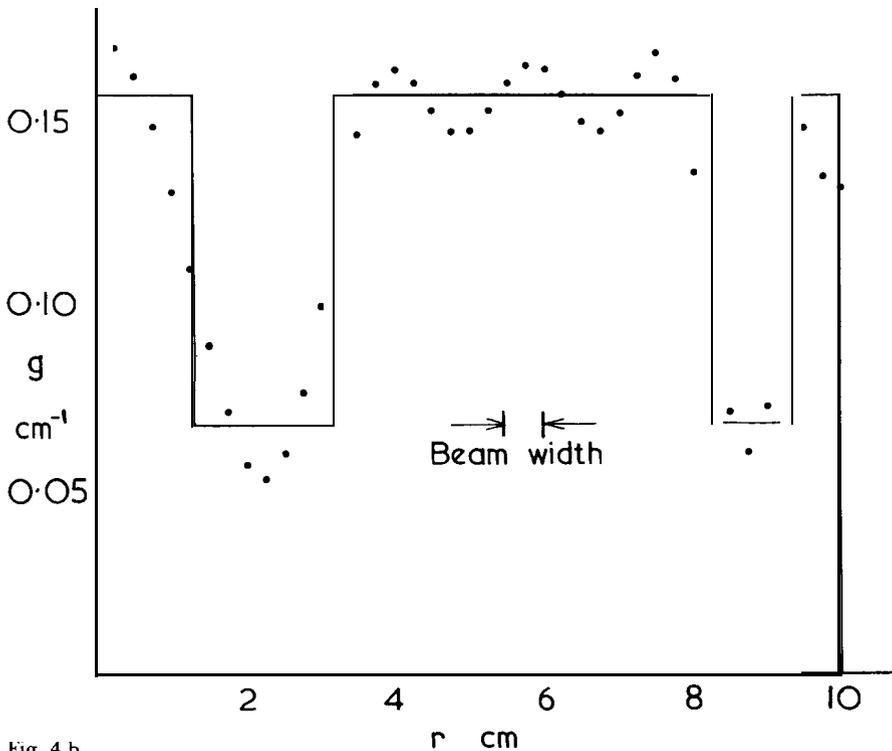


Fig. 4 b

learnt that this problem had come up in statistics in 1936 at the Department of Mathematical Statistics here in Stockholm where it was solved by Cramer and Wold (4). I also learnt of Bracewell's work (5) in radioastronomy which produced the same form of solution as Cramer and Wold had found, of De Rosier and Klug's (6) (1968) work in electron microscopy, and of the work of Rowley (7) (1969) and Berry and Gibbs (1970) in optics. Last, but by no means least, I first heard of Hounsfield and the EMI-scanner, about which you will be hearing shortly. Since 1972 I have been interested in a number of problems related to the CT-scanning problem or stemming from it, and I would like to tell you about some of them.

As I mentioned earlier the solution the Radon's problem which I found was in terms of integrals which extend from the radius at which the absorption coefficient is being found to the outer edge of the sample. What this means is that if one wishes to find the absorption coefficient in an annulus one needs only data from lines which do not intersect the hole in the annulus. This is known as the Hole Theorem and it is an exact mathematical theorem which can be proved very simply. However, as I have mentioned, when one applies my integrals to real data containing noise, the noise from the outer observations is amplified badly as one goes deeper and deeper into the sample, so the integrals as they stand are not useful in practice. This process of working from the outside in is known as "peeling the onion", and several algorithms for peeling the onion have been tried. All show the same feature, namely, that noise from the outer data propagates badly as one goes deeper and deeper into the sample. However, to the best of my knowledge, no one has produced a closed form solution to Radon's problem which includes the Hole Theorem and from which it can be *proved* that noise from the outside *necessarily* propagates badly into the interior. (Doyle and I published a closed form solution to Radon's problem which turned out to be wrong. A retraction will be published soon.) (9)

A solution containing the hole theorem for which noise did *not* propagate into the interior would be extremely valuable for CT-scanning. First it would reduce the dose administered to a patient by not passing X-rays through a region of no interest, i.e. the hole. Second it could be used to avoid regions in which sharp changes in absorption coefficient occur. These sharp changes necessarily produce overshoots which cloud the final image. In an extreme case consider a patient who by accident or by some medical procedure has in his body a piece of metal. This distorts a CT scan of a plane containing the metal beyond recognition. The hole theorem could be used to avoid the piece of metal. Less extremely, an interface between bone and soft tissue introduces overshoots which distort the image of the soft tissue. The hole theorem could be used to avoid the bone and the distortion.

Here then is a question which needs an answer. An unfavorable answer would leave us no worse off than we now are. A favorable answer would result in an improvement of imaging.

In my 1963 paper I had mentioned, but not very favorably, that protons could be used for scanning instead of X-rays. In 1968 my friend and colleague Andreas Koehler started producing some beautiful radiograms (10) using protons, and thinking about his work removed some of the doubts I had previously had about using protons for tomography.

The distinction between protons and X-rays is illustrated in Fig. 5. The number-distance curve for X-rays is the familiar exponential curve. For protons, there is a slow decrease caused by nuclear absorption followed by a very sharp fall-off. Most of the protons penetrate a distance into the material then all that are left stop in a short distance at what is called the range of the protons. It is this sharp fall-off which makes protons very sensitive to small variations in thickness or density along their paths. A comparison of the two curves at the range is a rough measure of this sensitivity to thickness or density changes. Others were thinking about the same thing and Crowe et al (11) at Berkeley produced a tomogram using alpha-particles in 1975, using the not inconsiderable arsenal of equipment at a high-energy physics laboratory. Koehler and I (12) made a tomogram of a simple circularly symmetrical sample to show two things:

1. that very simple equipment could be used, and
2. that we could easily detect 1/2% density differences with that equipment

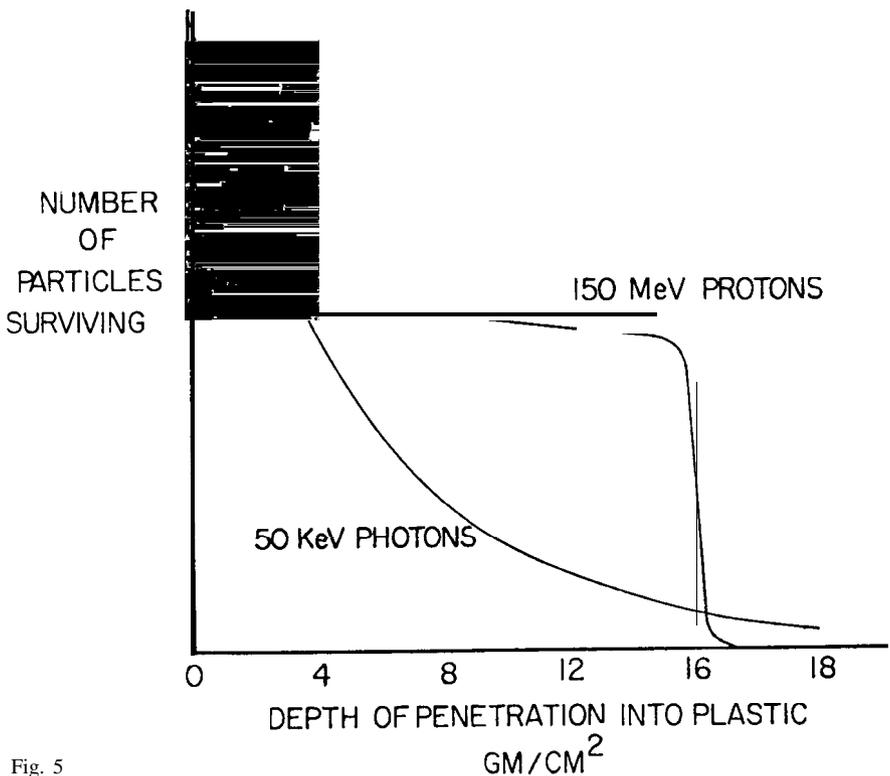


Fig. 5

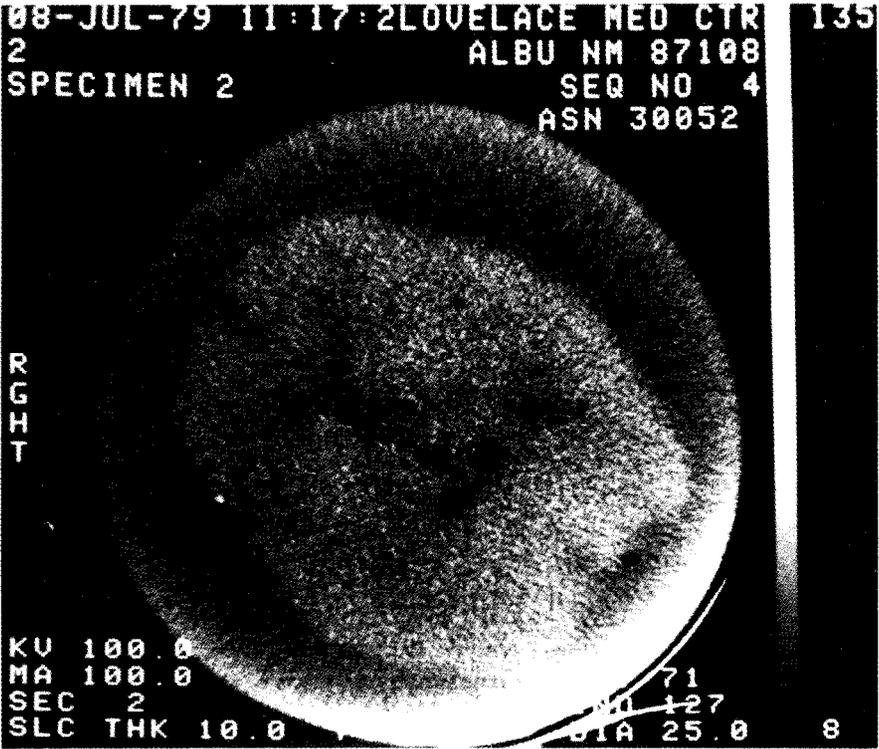


Fig. 6 a

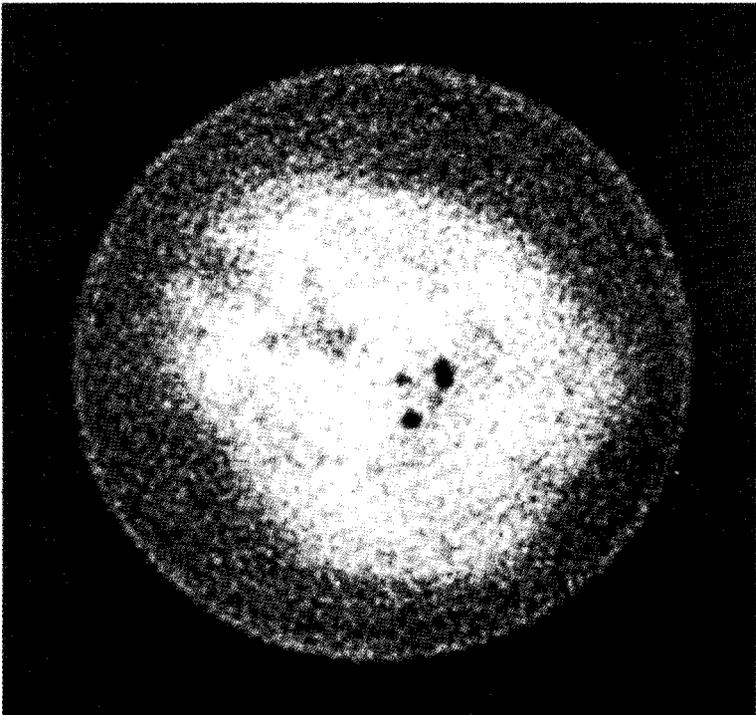


Fig. 6 b

As it happened, we were able to detect density differences of about 0.1% or less caused by machining stresses in our Lucite sample and diffusion of water into and out of the Lucite (13). More recently this work has been continued by Hanson and Steward (14) and their co-workers at Los Alamos, and I am able to show you some of their results. They have recently made tomograms of human organs. Fig. 6a is an X-ray scan of a normal brain made with a A2020 scanner at a dose of about 9 rad. Dr. Steward (14) has observed and made density measurements which show that water and electrolyte are lost by white tissue within four hours after death so density differences between grey and white matter disappear. Fig. 6b is a proton scan of the same sample made at a dose of 0.6 rad, and it is roughly as good as the X-ray scan. Figs. 7a and b show, respectively, an X-ray scan at a dose of 9 rad and a proton scan at 0.6 rads of a heart with a myocardial infarction fixed in formalin. According to Dr. Steward, fixing in formalin does not alter the relative stopping powers of the tissues of the heart. Again the pictures are of about the same quality. The time for taking the data was about an hour, but the LAMF accelerator is pulsed and is on for only 6% of the time. A continuous beam would have produced the same data in about four minutes - roughly the same time as the original EMI-scanner. I would like to acknowledge my indebtedness to Hanson and Steward *et al.* for making their results available to me.

Why use protons? First there is the question of dose. The dose in Hanson and Steward's work about 0.6 rad as distinct from to 9 rads required for the X-ray scans. This is in agreement with theoretical calculations (15) that the dose required for proton scans is five to ten times less than for X-ray scans for the same amount of information. Second, the mechanism by which protons are brought to rest is different from the mechanisms by which X-rays are absorbed or scattered, so one ought to see different things by using the different radiations, particularly the distribution of hydrogen. Koehler and I have indeed found a reversal of contrast between proton and X-ray scans on two occasions, and this is expected from theory.

You know how some people fuss about the high cost of CT-scanners, so you can imagine what they would say if one suggested that the X-ray tube in the scanner should be replaced by a much more expensive cyclotron! Of course they would be right, but only if proton scanning is viewed in the narrow context of diagnostics. Instead of looking at the number-distance curve which I showed when comparing X-rays and protons, let us look at the ionization-distance curve which is the thing that matters for therapy. This is shown in Fig. 8. You will see that the curve is peaked (the Bragg peak) near the end of the range and then falls off very sharply, so that there is virtually no ionization beyond the end of the range. Both of these facts are of importance in therapy as was pointed out by R. R. Wilson (16) in 1946, and as has been shown by treatments of a number of different conditions at Uppsala, Harvard and Berkeley. In fact, protons are far superior to X-rays for the treatment of some conditions. So one can

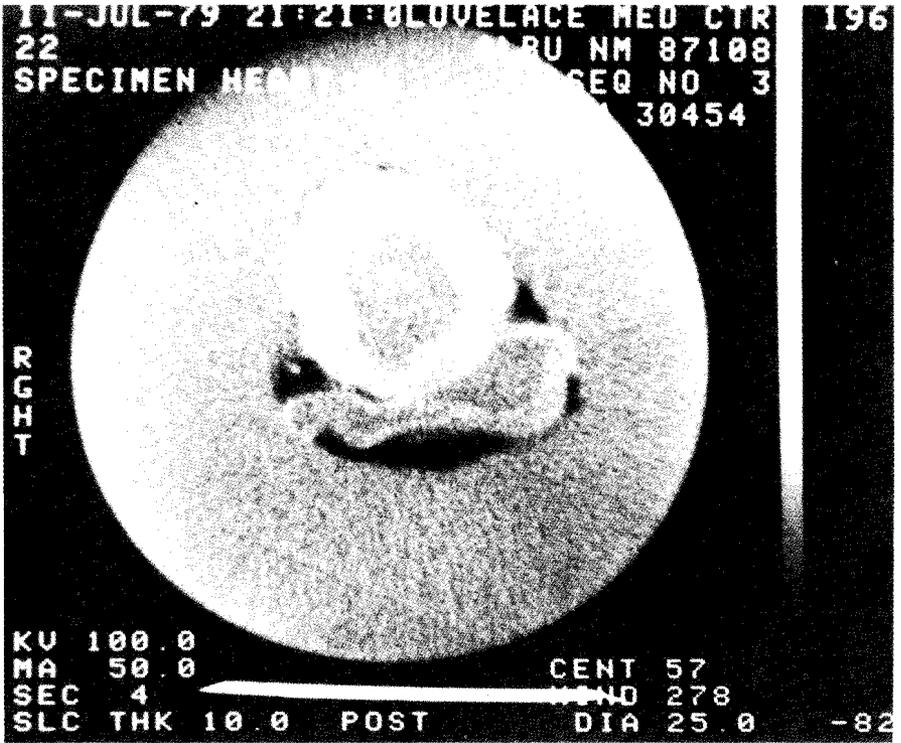


Fig. 7 a

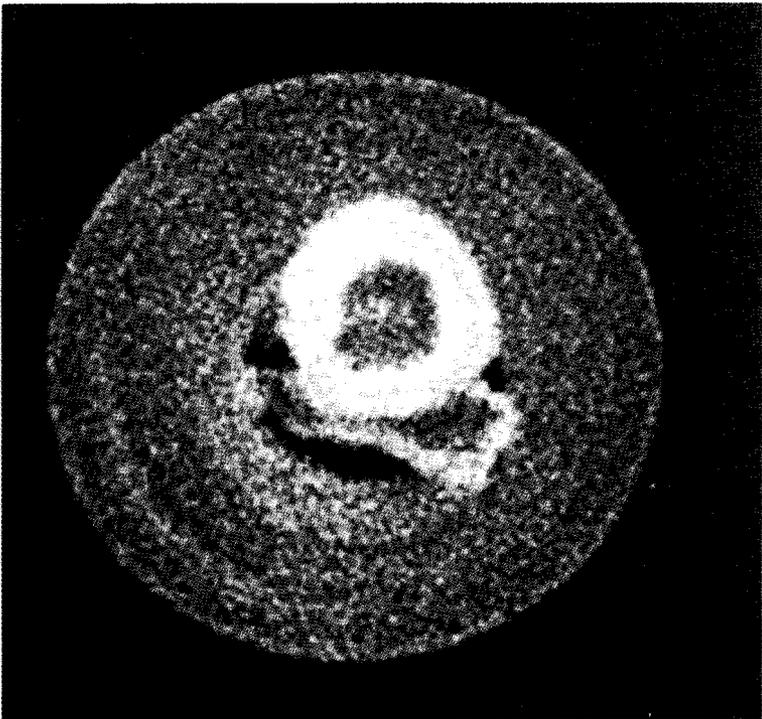


Fig. 7 b

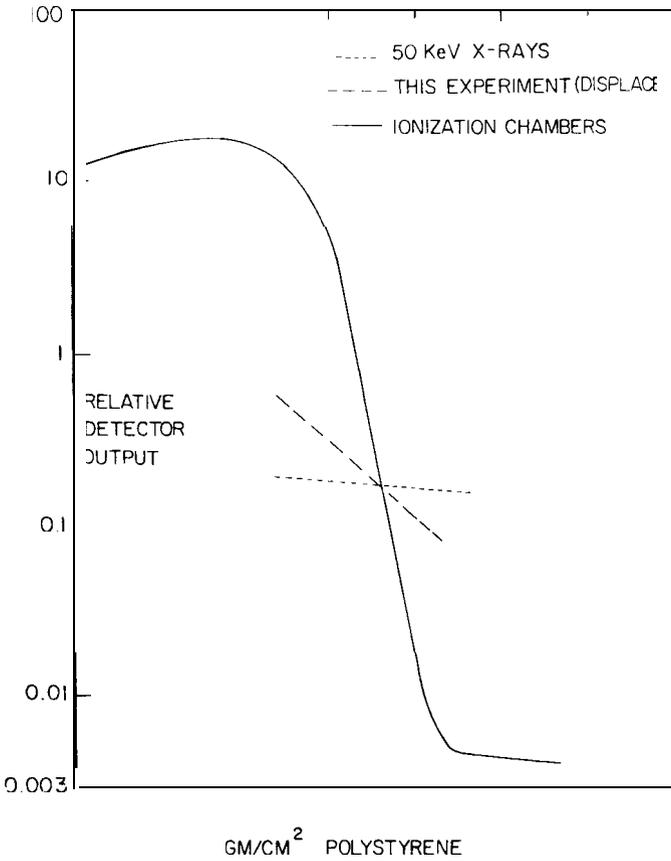


Fig. 8

envisage a large metropolitan area as having a 250 MeV proton accelerator with a number of different ports, say ten, at which patients could be treated simultaneously (17). If one of these ports was devoted to proton tomography, the marginal cost of such tomography would not be great. In exploring these possibilities it is essential that diagnostic radiologists and therapeutic radiologists work together.

Most recently I have been interested in some mathematical topics which I would like to discuss without mathematical details. I have presented Radon's problem in the form in which it is usually given in CT-scanning, namely, how does one recover a function in a plane given its line integrals over all lines in the plane. This can be generalized to three-dimensions as can be seen in Fig. 9. This shows a sphere in which a function is defined, and a plane intersecting that sphere. Suppose that the given information is the integrals of the function over all planes intersecting the sphere, then the question is "Can the function be recovered from these integrals?" Radon provided an affirmative answer and a formula for finding the function. It turns out that this formula is simpler than in the case of two-dimensions, and it has applications in imaging using Nuclear Magnetic

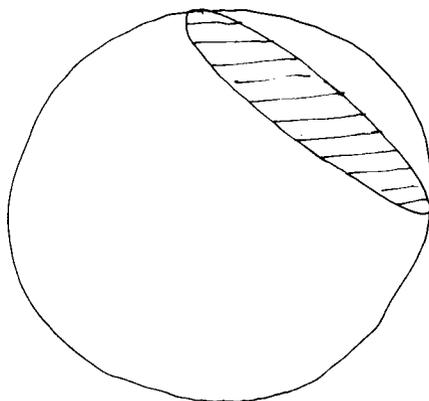


Fig. 9

Resonance (N.M.R.) rather than X-rays. The generalization to n -dimensions follows in a straightforward way for Euclidean space and was given by Radon. The few mathematicians who knew of Radon's work before the nineteen sixties applied his results to the theory of partial differential equations. They also generalized his results to integration over circles or spheres of constant radius and to certain ellipsoids. Generalizations to spaces other than Euclidean were made, for example, by Helgason (18) and Gelfand *et al* (19) in the sixties.

In my 1964 paper I gave a solution to the problem of integrating over circles of variable radius which pass through the origins as shown in Fig. 10. Recently Quinto and I (20) have given much more detailed results on this problem generalized to n -dimensional Euclidean space, and we have applied these results to obtain some theorems about the solutions of Darbox's partial differential equation. There is an intimate connection

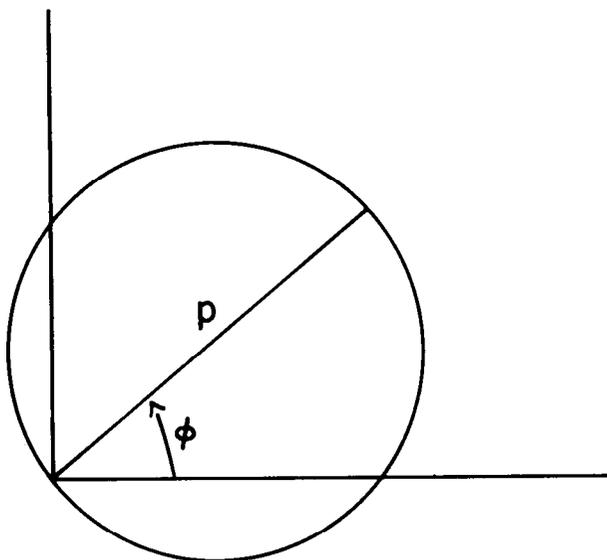


Fig. 10

between our results and Radon's results, and we are presently attempting to find more general results which relate the solutions of Radon's problem for a family of surfaces to the solution of Radon's problem for another family of surfaces related to the first in a particular way.

What is the use of these results? The answer is that I don't know. They will almost certainly produce some theorems in the theory of partial differential equations, and some of them may find application in imaging with N.M.R. or ultrasound, but that is by no means certain. It is also beside the point. Quinto and I are studying these topics because they are interesting in their own right as mathematical problems, and that is what science is all about.

Of the many people who have influenced me beneficially, I shall name only a few. The late Professor R. W. James F.R.S. taught me a great deal, not only about physics, when I was a student and a young Lecturer in his Department in Cape Town. Andreas Koehler, Director of the Harvard Cyclotron Laboratory, has provided me with friendship, moral support, and intellectual stimulation for over twenty years. On the domestic side are my parents, now deceased, and my immediate family. My wife Barbara and our three children have not only put up with me, they have done so in a loving and supportive way for many years. To these people and others unnamed I shall be grateful to the end of my life.

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